Applications of Bitstrings for Entanglement in the Quantum Ising Model

Quantum Technologies for Fundamental Physics

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Background

- An arbitrary state vector is

$$|\psi\rangle = \sum_{\{n\}} c_{\{n\}} |\{n\}\rangle \tag{1}$$

- Where, for N qu$d$its with d states in the local Hilbert space, the computational basis has d^N states of the form

$$|\{n\}\rangle \equiv |n_0, n_1, \ldots, n_{N_q-1}\rangle, \tag{2}$$

- With this basis for a partitioned system, the vector can be split according to

$$|\{n\}_{ABC\ldots}\rangle = |\{n\}_A\rangle |\{n\}_B\rangle |\{n\}_C\rangle \ldots, \tag{3}$$

Background

- A state $|\{n\}\rangle$ is observed with probability

$$p_{\{n\}} = |c_{\{n\}}|^2 \tag{4}$$

- A marginal probability can be defined

$$p_{\{n\}_R} \equiv \sum_{\{n\}_{\bar{R}}} p_{\{n\}_R\{n\}_{\bar{R}}} \tag{5}$$

Entropy Definitions

- A density matrix for the whole system is written as

$$\rho_{R\bar{R}} = |\psi\rangle\langle\psi|. \tag{6}$$

- So, when diagonal, becomes

$$\rho_{R\bar{R},diag} = \sum_{\{n\}} |c_{\{n\}}|^2 |\{n\}\rangle\langle\{n\}|. \tag{7}$$

- To get the bipartite entanglement, we need a reduced density matrix

$$\rho_R = \mathrm{Tr}_{\bar{R}}\, \rho_{R\bar{R}}, \tag{8}$$

- Which gives the bipartite entanglement

$$S_R^{vN} \equiv -\,\mathrm{Tr}(\rho_R \ln(\rho_R)) = -\sum_r \lambda_r \ln \lambda_r \tag{9}$$

Entropy Definitions

- We can also define the Shannon entropy, using our probabilities, as

$$H_{R\bar{R}} \equiv -\sum_{\{n\}} p_{\{n\}} \ln(p_{\{n\}}). \tag{10}$$

- Using the previous marginal probabilities, we can get a reduced Shannon entropy

$$H_R \equiv -\sum_{\{n\}_R} p_{\{n\}_R} \ln p_{\{n\}_R}. \tag{11}$$

- This allows us to get the mutual information of the system with the "experimental" probabilities

$$I_{R\bar{R}} \equiv H_R + H_{\bar{R}} - H_{R\bar{R}}. \tag{12}$$

Finding the Critical Point

- We can use the following quantities[3] to identify critical points of the system[4].
- Strong Subadditivity

$$S_{strong} = S_{AB} + S_{BC} - S_B - S_{ABC} \geq 0 \tag{13}$$

- Weak Monotonicty

$$S_{weak} = S_{AB} + S_{BC} - S_A - S_C \geq 0 \tag{14}$$

- Which are interpolated with a factor related to the individual parts size, η, to give the quantity

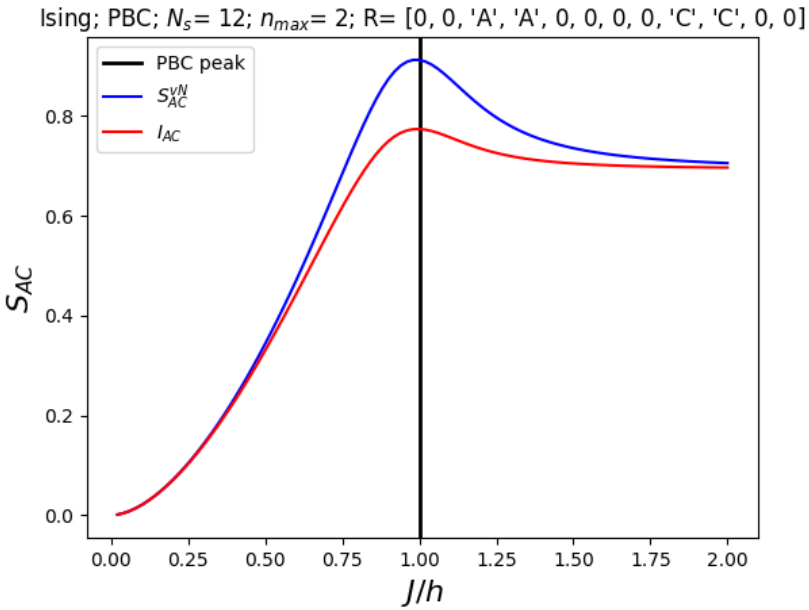
$$S_\Delta := (S_{AB} + S_{BC}) - \eta(S_A + S_C) - (1-\eta)(S_B + S_{ABC}) \tag{15}$$

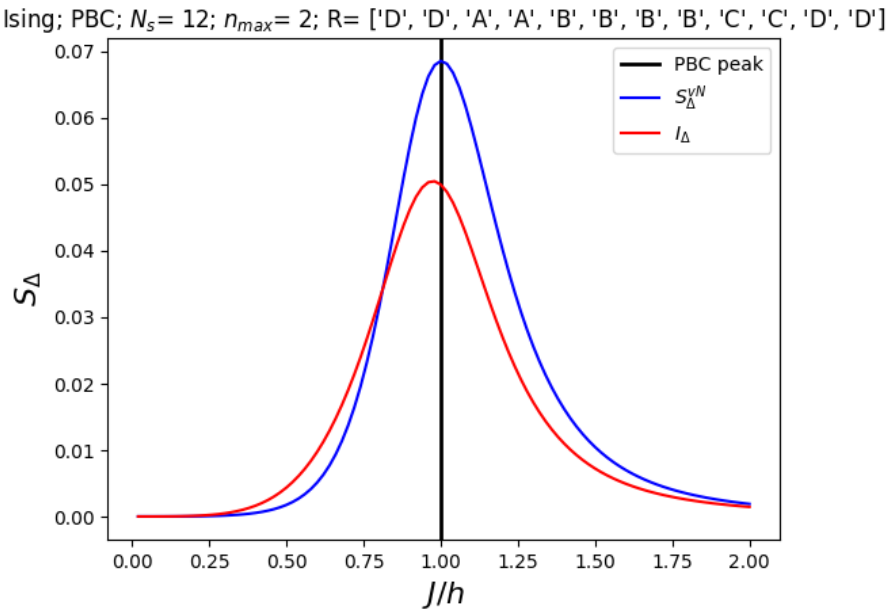
[3] A.S. Holevo. T.-C. Lin and J. McGreevy, "Conformal Field Theory Ground States as Critical Points of an Entropy Function," Phys. Rev. Lett. 131 (2023), **25** 251602, arXiv:2303.05444.

[4] Z. Ozzello and Y. Meurice, "Multipartite entanglement for d-level systems in 1+1D,"Phys. Rev. D **113** (2026), no.7, 074504, arXiv:2507.14422.

Transverse Field Ising Model (TFI)

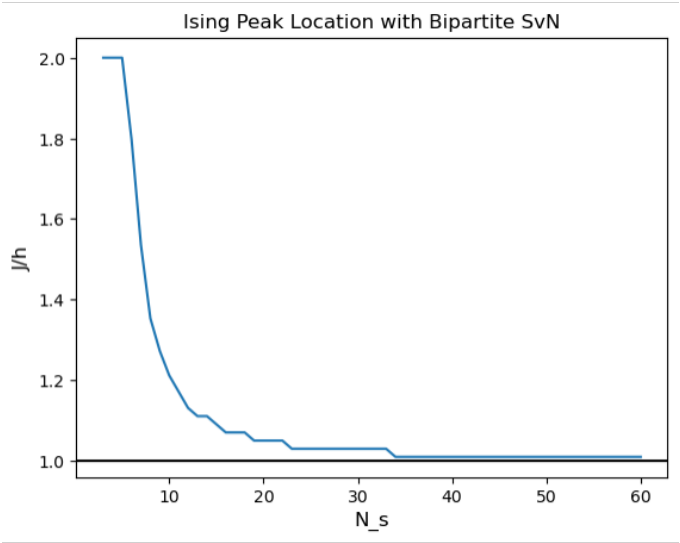
$$H_{TFI} = -h \sum_{i=1}^{N_s} \sigma_i^z - J \sum_{i=1}^{N_s} \sigma_i^x \sigma_{i+1}^x$$

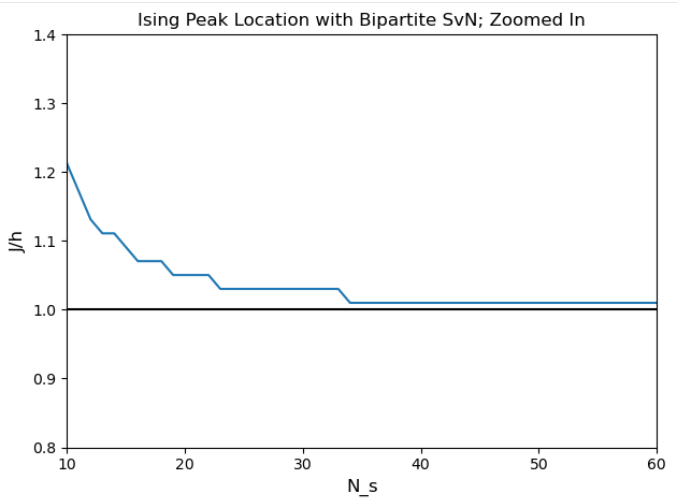




Left: Bipartite entanglement; Right: S_{Δ} combination

Why not Use Bipartite Peak?





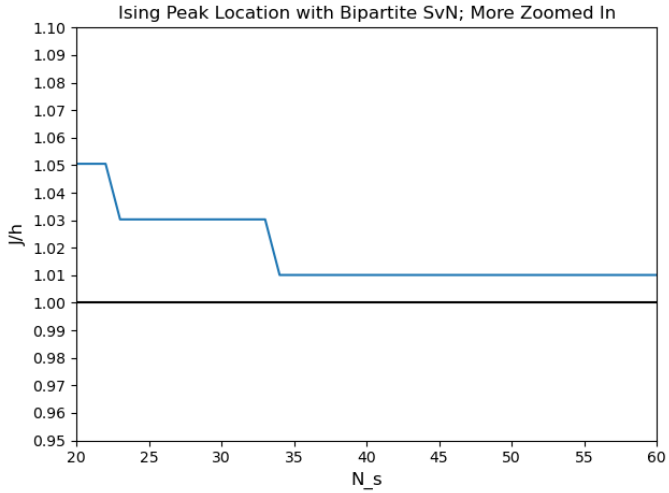
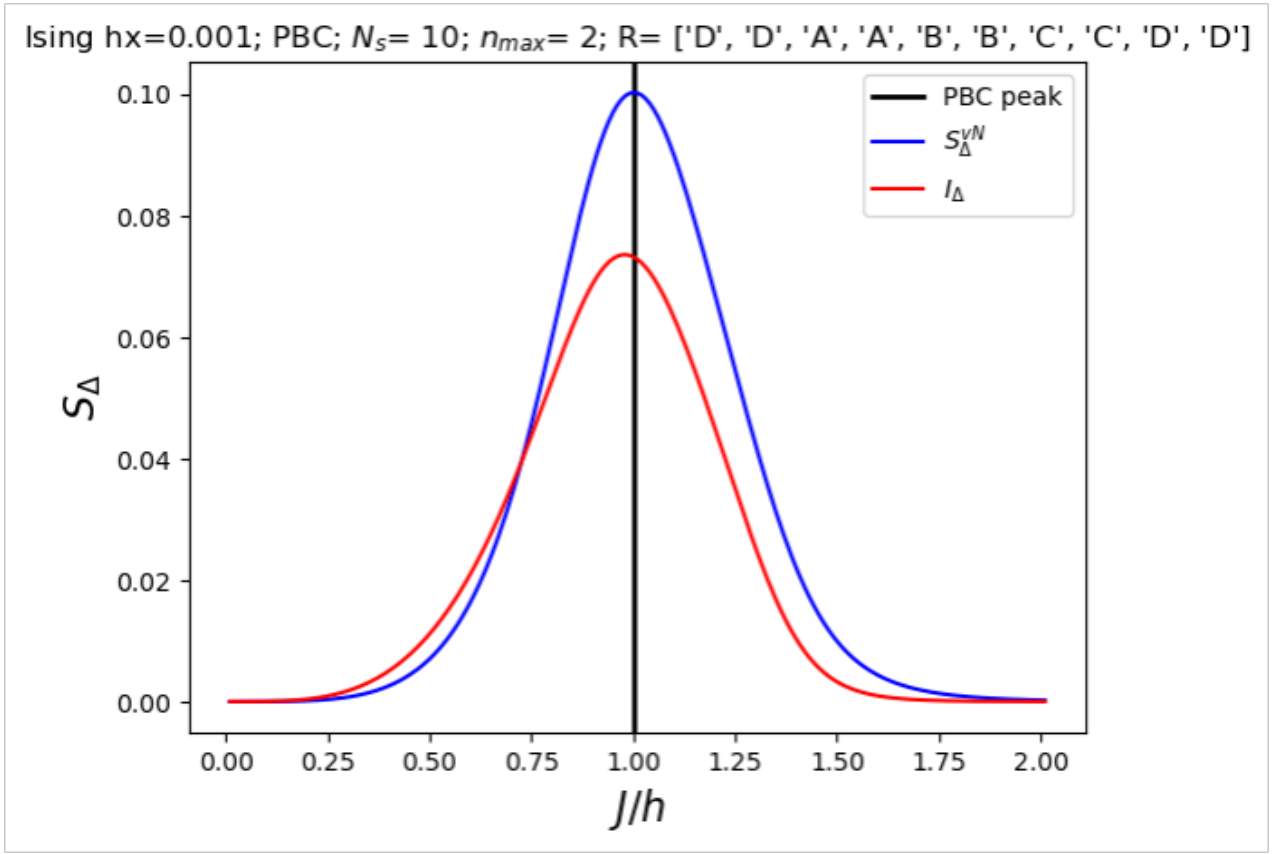


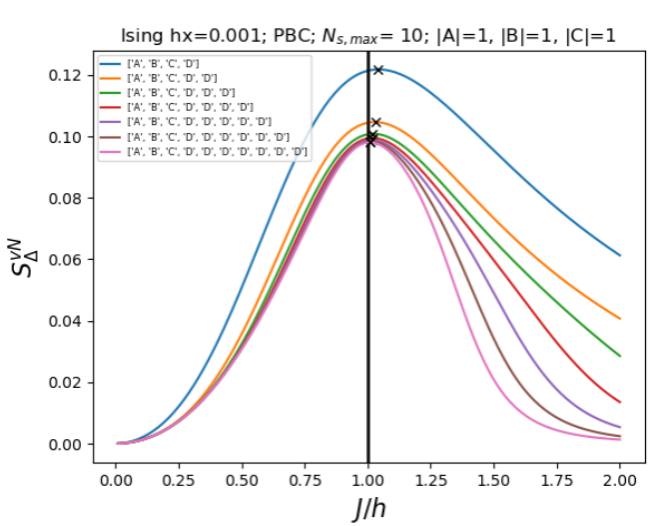
Figure: Peak position, in J/h, of half-system bipartite entanglement

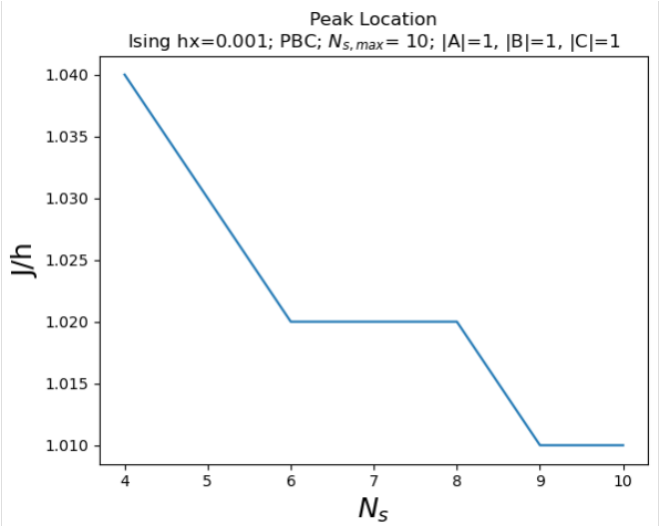
What about with a Longitudinal Field?

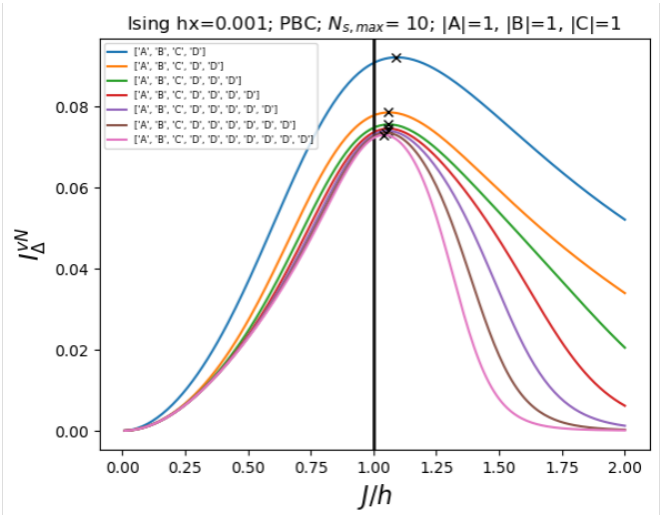
$$H_{LFI} = -h_z \sum_{i=1}^{N_s} \sigma_i^z - h_x \sum_{i=1}^{N_s} \sigma_i^x - J \sum_{i=1}^{N_s} \sigma_i^x \sigma_{i+1}^x$$

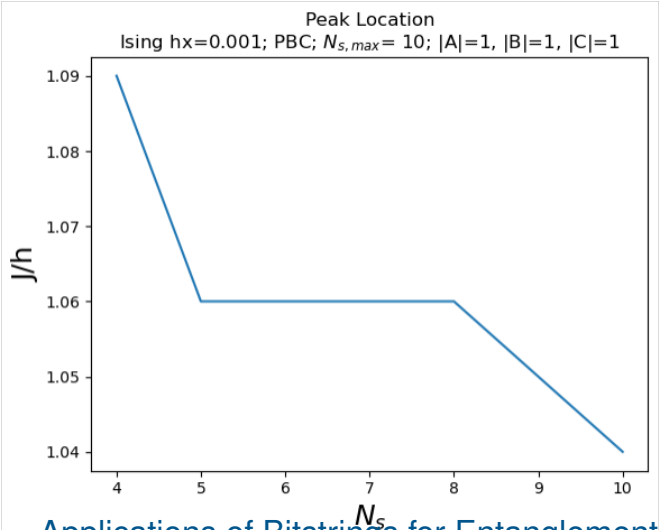


Movement of Peaks(with Longitudinal Field)









Closing Comments

- Read more here:
  Z. Ozzello and Y. Meurice, “Multipartite entanglement for d-level systems in 1+1D,” Phys. Rev. D **113** (2026), no.7, 074504, arXiv:2507.14422.
- Feel free to reach out and email me at zane-ozzello@uiowa.edu
- Questions?
- Thanks for listening!