Explroing Quantum Algorithms

for Combinatorial Problems at Colliders

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The Problem

$2 \to 2$ Collision

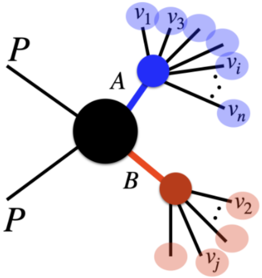

Figure 1: Not a frog seen from above.
Or Salad Fingers
From 2111.07806 [1]

- A binary classification problem
- There are $\mathcal{O}(2^n)$ possible classifications.
- What happens if $n \geq 1000$ or so?

Example: $pp \to t\bar{t}$

Two dominant decay modes for t:

- $t \to W^+ b \to \ell \nu_\ell b$
  - ✓ Cleaner – only one jet
  - × Missing momentum
- $t \to W^+ b \to q\bar{q}b$
  - × Messier – 3 quarks
  - ✓ All momentum accounted for

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We will look at this decay

What are they?

- A qubit is a two-dimensional vector: $\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$
- Describe it as two basis vectors: $\alpha\begin{pmatrix}1 \\ 0\end{pmatrix} + \beta\begin{pmatrix}0 \\ 1\end{pmatrix}$
- Use fancy QM notation: $|\psi\rangle = \alpha\ |0\rangle\ + \beta\,|1\rangle$
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- You change vector with matrices.
- Quantum 1-qubit "gate" is a 2x2 matrix:

$$U\,|\psi\rangle = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \text{new state!}$$

Two Qubits

Two qubits are a four-dimensional vector:

$$|\psi\rangle = \alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle$$

$$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ |10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

N Qubits

Initial state

Behold, an N-qubit circuit:

$$\underbrace{U_{2^N \times 2^N} \quad |0\rangle^{2^N}}$$

Final State

Variational Quantum Algorithms (VQA)

1. Create a cost function $C(\boldsymbol{\theta})$ that you want to minimize,
   - (In a physics context, think of the Hamiltonian, action, entropy, etc.)
2. And parameterized quantum circuit ansatz that produces state $|\boldsymbol{\theta}\rangle$ to find $\langle\boldsymbol{\theta}|C|\boldsymbol{\theta}\rangle$.
3. A classical optimizer changes values of $\boldsymbol{\theta}$ to lower cost function.
4. Repeat until the minimum is found

Just like a neural network but the network is replaced by a quantum circuit

What can we minimize for $t\bar{t}$?

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where $P_1 = \sum$(particles' p assigned to t)

$P_2 = \sum$(particles' p assigned to $\bar{t}$)

4-momentum squared equals mass squared:

$$E^2 = \mathbf{p}^2 + m^2 \Longrightarrow m^2 = E^2 - \mathbf{p}^2 = p^\mu p_\mu = p^2$$

Quantum Approximation Optimization Algorithm (QAOA)

1. Create ansatz from classical H_P: A Hamiltonian operator $\hat{H}_P$.
2. Exploit the adiabatic theorem to evolve to ground state of $\hat{H}_P$: $|\psi\rangle$.
3. Take expectation value: $\langle\psi|\hat{H}_P|\psi\rangle$.
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Wait... what parameters?

Adiabatic Theorem

Exploiting (and enjoying) the Adiabatic theorem:

$$\hat{H}(t) = \left(1 - \frac{t}{T}\right) \hat{H}_M + \frac{t}{T} \hat{H}_P$$

$H = H_M$ at early times and $H = H_P$ at late times.

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Simple and solvable

Ground state is your answer!

Let's say for no particular reason $\hat{H}_M = \sum_{i=1}^{n} \sigma_i^x$

Plan of Action

Schrödinger equation:

$$\hat{H}\left|\psi\right\rangle = i\frac{\partial}{\partial t}\left|\psi\right\rangle \quad \Longrightarrow \quad \left|\psi\right\rangle = e^{-i\hat{H}t}\left|\psi_0\right\rangle$$

If $t = T$, then $\left|\psi_0\right\rangle$ is the ground state of $\hat{H}_M$, then $\left|\psi\right\rangle$ is the ground state of $\hat{H}_P$.

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If $t = T$, then $\left|\psi_0\right\rangle$ is the ground state of $\hat{H}_M$, then $\left|\psi\right\rangle$ is the ground state of $\hat{H}_P$. *But how?*

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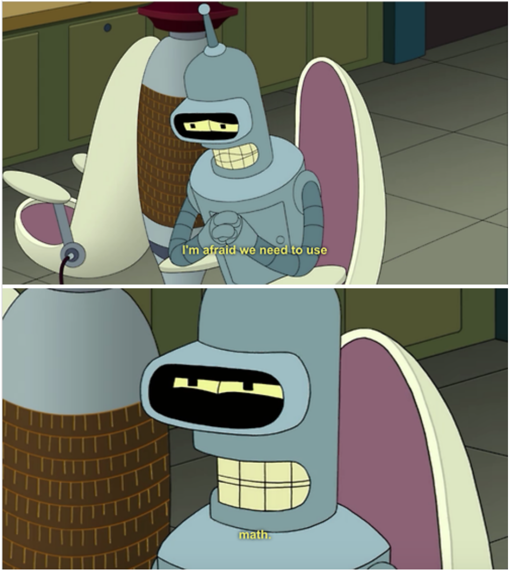

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We did it!

$$|\boldsymbol{\gamma}, \boldsymbol{\beta}\rangle = \prod_{j=1}^{p} \exp\left[-i\beta_j \hat{H}_M\right] \exp\left[-i\gamma_j \hat{H}_P\right] |+\rangle$$

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$$H_P = (P_1^2 - P_2^2)^2 = \sum J_{ij} s_i s_j \Longrightarrow \hat{H}_P = \sum J_{ij} \sigma_i^z \sigma_j^z$$

This is just the Ising model sans an external field

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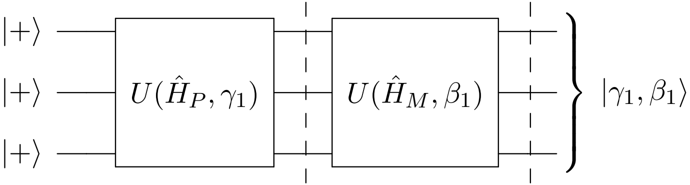
Small steps of time-evolutions...

- $\exp\left[-i\beta_j \hat{H}_M\right]$: An x rotation of all n qubits separately
- $\exp\left[-i\gamma_j \hat{H}_P\right]$: A zz rotation on all qubit pairs

The Circuit

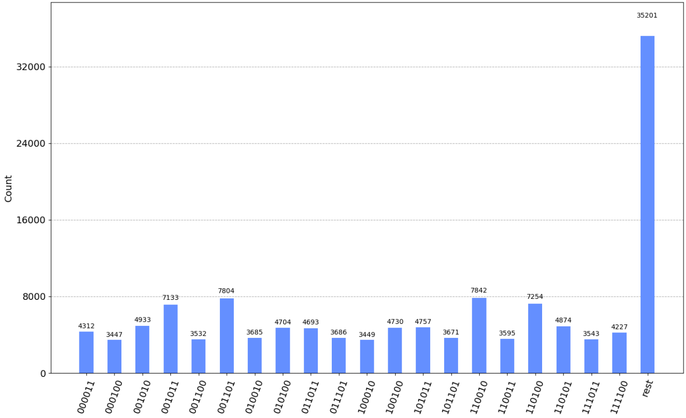
For 3 qubits and a depth of $p = 1$:



Then feed $\langle \gamma_1, \beta_1 | \hat{H}_P | \gamma_1, \beta_1 \rangle$ into a classical optimizer and repeat.

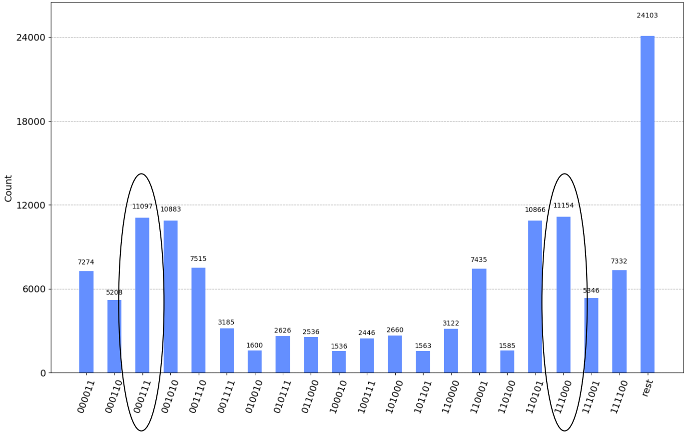
Results

Depth of $p = 1$:



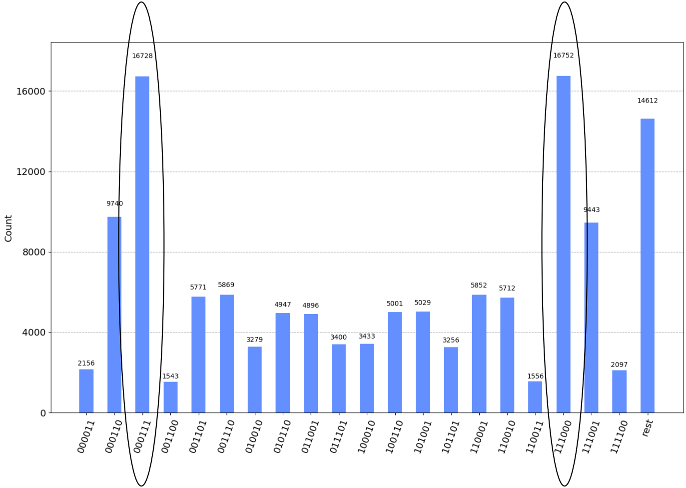
Results

Depth of $p = 5$:



Results

Depth of $p = 10$:



Pros & Cons

Pros

- Finds the global minimum (as $p \to \infty$)
- Scales less than exponentially
- QC is very popular – things will only get better

Cons

- Can get stuck in local minima
- May need large p
- Quantum advantage only for large n
- Gates are noisy – NISQ era for a reason

Future

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- Noise go down, qubits go up.

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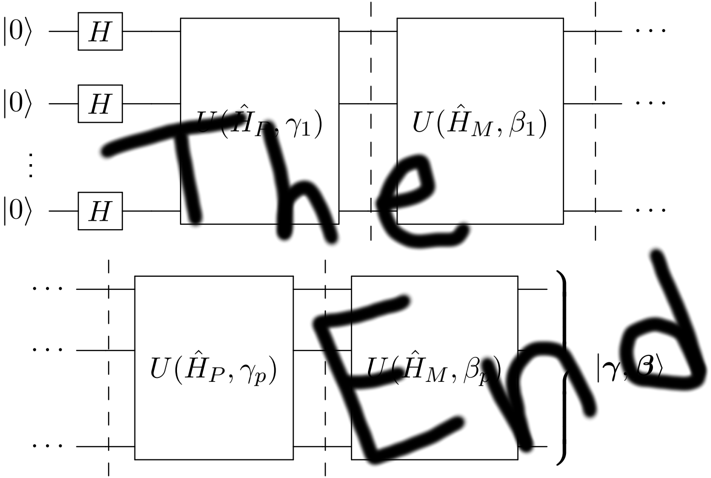
- Noise go down, qubits go up.
- But also QAOA isn't the only contender in town, e.g. Feedback-Based ALgorithm for Quantum OptimizatioN (FALQON)[1]

[1] 2103.08619 [3]



$$\min_{\boldsymbol{\gamma},\boldsymbol{\beta}} \langle \boldsymbol{\gamma}, \boldsymbol{\beta} | H_P | \boldsymbol{\gamma}, \boldsymbol{\beta} \rangle$$

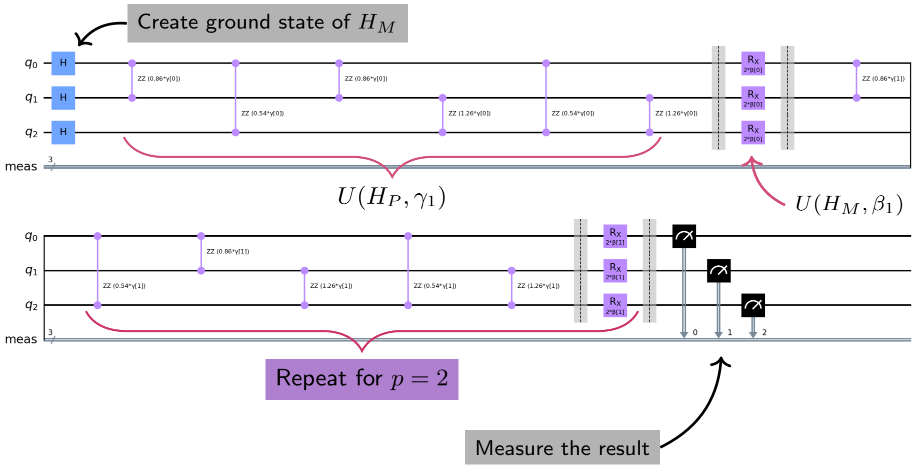
References

[1] Minho Kim et al. *Leveraging Quantum Annealer to identify an Event-topology at High Energy Colliders*. 2021. arXiv: 2111.07806 [hep-ph].

[2] Edward Farhi, Jeffrey Goldstone, and Sam Gutmann. *A Quantum Approximate Optimization Algorithm*. 2014. arXiv: 1411.4028 [quant-ph].

[3] Alicia B. Magann et al. "Feedback-Based Quantum Optimization". In: *Physical Review Letters* 129.25 (Dec. 2022). DOI: 10.1103/physrevlett.129.250502. arXiv: 2103.08619 [quant-ph]. URL: https://doi.org/10.1103%2Fphysrevlett.129.250502.

The Quantum Circuit in Detail



Deriving the Ising Model Form

Our Hamiltonian is $H_P = (P_1^2 - P_2^2)^2$ where $P_1 = \sum p_i x_i$ and $P_2 = \sum p_i(1 - x_i)$ where $x_i = 0$ if final particle i is assigned to particle 1 and 0 otherwise. Then let $x_i = (1 + s_i)/2$ so $s_i = \pm 1$.

$$
\begin{aligned}
P_1^2 + P_2^2 &= \frac{1}{4}\sum_{ij} P_{ij}(1 + s_i)(1 + s_j) - \frac{1}{4}\sum_{ij} P_{ij}(1 - s_i)(1 - s_j) \\
&= \frac{1}{4}\sum_{ij} P_{ij}(1 + s_i s_j + 2s_i) - \frac{1}{4}\sum_{ij} P_{ij}(1 + s_i s_j - 2s_i) \\
&= \sum_{ij} P_{ij} s_i
\end{aligned}
$$

where $P_{ij} = p_i \cdot p_j$ is the dot product of 4-momentum.

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$$(P_1^2 + P_2^2)^2 = \sum_{ijk\ell} P_{ij} P_{k\ell} s_i s_k$$

$$= \sum_{ij} J_{ij} s_i s_j \qquad \text{where} \qquad J_{ij} = \sum_{k\ell} P_{ik} P_{j\ell}$$

where $P_{ij} = p_i \cdot p_j$ is the dot product of 4-momentum.